Newtonian mechanics - falling body problems

We will show how the physics of the falling body problem leads naturally to a differential equation.

Consider a mass m falling due to gravity. We orient coordinates to that downward is positive and let $x(t)$ denote the distance fallen at time t.

We *assume* only two forces act: the force due to gravity, F_{grav}, and the force due to air resistence, F_{res}. In other words, we assume that the total force is given by

$$F_{total} = F_{grav} + F_{res}.$$

We know that $F_{grav} = mg$, where g is the gravitational constant. We assume, as is common in physics, that air resistence is proportional to velocity: $F_{res} = -kv = -kx'(t)$, where $k \geq 0$ is a constant. Newton's second law tells us that $F_{total} = ma = mx''(t)$. Putting these all together gives $mx''(t) = mg - kx'(t)$, or

$$v'(t) + \frac{k}{m}v(t) = g. \tag{1}$$

This is the differential equation governing the motion of a falling body. Equation (1) can be solved by various methods: separation of variables or by integrating factors. If we assume $v(0) = v_0$ is given and if we assume $k > 0$ then the solution is

$$v(t) = \frac{mg}{k} + (v_0 - \frac{mg}{k})e^{-kt/m}. \tag{2}$$

In particular, we see that the limiting velocity is $v_{limit} = \frac{mg}{k}$.

Example: A parachutist weights 100 kgs (with chute). The chute is released 30 seconds after the jump from a height of 2000 m. The force due to air resistence is given by $\vec{F}_{res} = -k\vec{v}$, where

$$k = \begin{cases} 15, & \text{chute closed,} \\ 100, & \text{chute open.} \end{cases}$$

Find

(a) the distance and velocity functions during the time when the chute is closed (i.e., $0 \leq t \leq 30$ seconds),

(b) the distance and velocity functions during the time when the chute is open (i.e., $30 \leq t$ seconds),

(c) the time of landing,
(d) the velocity of landing.
soln: Taking $m = 100$, $g = 9.8$, $k = 15$ and $v(0) = 0$ in (2), we find

$$v_1\,(t) = \frac{196}{3} - \frac{196}{3}\,e^{-\frac{3}{20}\,t}.$$

This is the velocity with the time t starting the moment the parachutist jumps. After $t = 30$ seconds, this reaches the velocity $v_0 = \frac{196}{3} - \frac{196}{3}e^{-9/2} = 64.607...$. The distance fallen is

$$\begin{aligned} x_1(t) &= \int_0^t v_1(u)\ du \\ &= \tfrac{196}{3}\,t + \tfrac{3920}{9}\,e^{-\frac{3}{20}\,t} - \tfrac{3920}{9}. \end{aligned}$$

After 30 seconds, it has fallen $x_1(30) = \frac{13720}{9} + \frac{3920}{9}e^{-9/2} = 1529.283...$ meters.
Taking $m = 100$, $g = 9.8$, $k = 100$ and $v(0) = v_0$, we find

$$v_2\,(t) = \frac{49}{5} + e^{-t}\left(\frac{833}{15} - \frac{196}{3}\,e^{-9/2}\right).$$

This is the velocity with the time t starting the moment the chute is opened (i.e., 30 seconds after jumping). The distance fallen is

$$\begin{aligned} x_2(t) &= \int_0^t v_2(u)\ du + x_1(30) \\ &= \tfrac{49}{5}\,t - \tfrac{833}{15}\,e^{-t} + \tfrac{196}{3}\,e^{-t}e^{-9/2} + \tfrac{71099}{45} + \tfrac{3332}{9}\,e^{-9/2}. \end{aligned}$$

Here's the graph of the velocity $0 < t < 50$. Notice how it drops at $t = 30$ when the chute is opened.

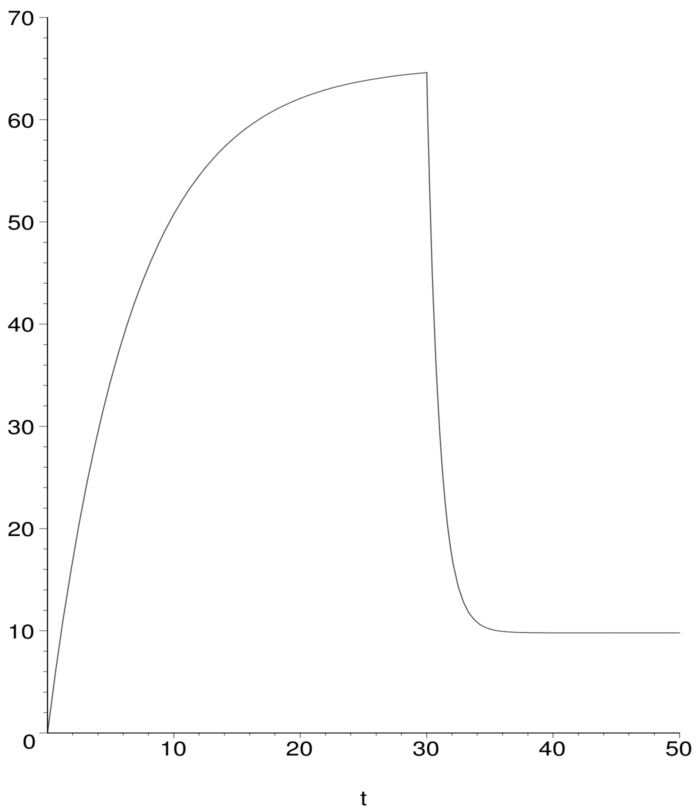

Figure 1: Velocity of falling parachutist.

Here's the graph of the distance fallen $0 < t < 50$. Notice how it slows down at $t = 30$ when the chute is opened.

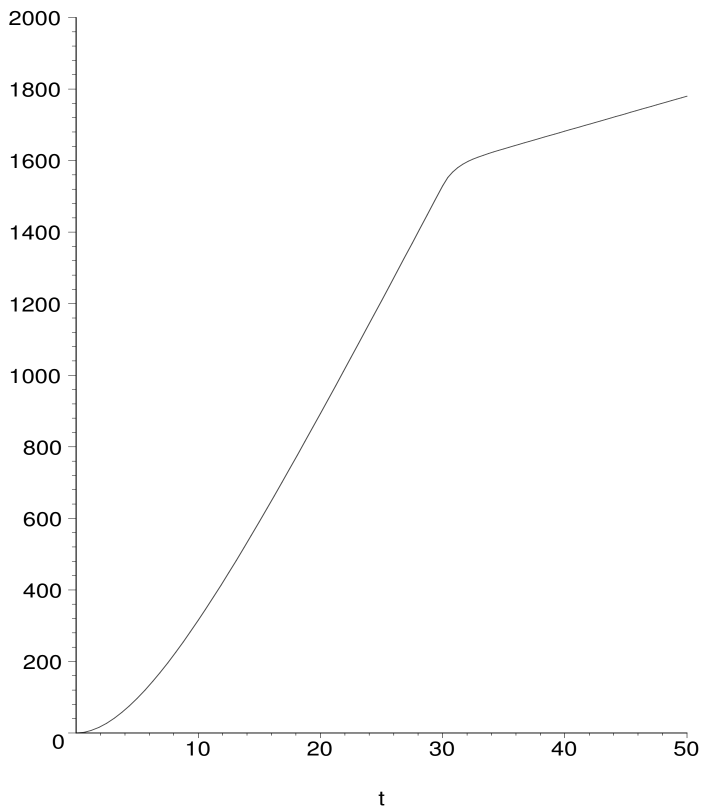

Figure 2: Distance fallen by a parachutist.

The time of impact is $t_{impact} = 42.4397....$ This was found numerically by solving $x_2(t) = 2000$.

The velocity of impact is $v_2(t_{impact}) = 9.8....$